An Efficient Technique for Identifying Limits in the Convolution of Random Variables

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Abstract

The convolution of two independent random variables plays a fundamental role in probability theory and signal processing. However, one of the most challenging aspects of this method is determining the correct integration limits, which often requires case-by-case analysis. This paper introduces a novel and systematic approach to finding these limits, making convolution calculations more efficient. The proposed technique is illustrated through a detailed example, demonstrating its effectiveness in simplifying limit identification.

1 Introduction

1.1 Background

The probability density function (PDF) of the sum of two independent continuous random variables X and Y is given by the convolution integral:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx.$$

One of the most challenging aspects of this integral is **determining the correct integration limits for** x. Traditional approaches require manually checking multiple cases, which can be error-prone and computationally inefficient.

1.2 Objective

This paper presents a new method to **systematically determine the integration limits** without trial and error. The technique is applied to an example to illustrate its efficiency and correctness.

2 Mathematical Foundation

We consider two independent random variables with the following PDFs:

$$f_X(x) = \begin{cases} \frac{1}{3}, & 0 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{4}, & -2 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

We define $W = X + Y$ and aim to find the probability density function $f_W(w)$.

2.1 Finding the Range of W

The minimum and maximum possible values of W are:

$$W_{\min} = \min(X) + \min(Y) = 0 + (-2) = -2.$$

$$W_{\max} = \max(X) + \max(Y) = 3 + 2 = 5.$$

Thus, W is defined over $-2 \leq W \leq 5$.

2.2 Identifying Integration Limits Using the New Technique

We rewrite Y in terms of X and W:

$$y = W - x.$$

Since Y is defined in $-2 < y < 2$, substituting $y = W - x$, we obtain:

$$-2 < W - x < 2.$$

Rearranging:

$$W - 2 < x < W + 2.$$

However, x is also constrained by $0 < x < 3$. The true integration limits for x come from the overlap of these two intervals:

$$\max(0, W - 2) < x < \min(3, W + 2).$$

3 Identifying Key Transition Points

The convolution function changes behavior at specific transition points. Using the **systematic comparison of left and right boundaries**, we solve:

3.1 Left Endpoint (Lower Limit of x)

$$\max(0, W - 2).$$

Setting $W - 2 = 0$, we get $W = 2$. This means for $W < 2$, the lower limit is 0, and for $W > 2$, it changes.

3.2 Right Endpoint (Upper Limit of x)

$$\min(3, W + 2).$$

Setting $W + 2 = 3$, we get $W = 1$. This means for $W > 1$, the upper limit is 3, but for $W < 1$, it is $W + 2$.

Thus, the convolution formula splits into three cases based on the values $W = 1$ **and** $W = 2$ as transition points.

4 Computation of $f_W(w)$ Using the New Limits

4.1 Case 1: $-2 < W < 1$

$$\int_0^{W+2} \frac{1}{3} \times \frac{1}{4} dx = \frac{W+2}{12}.$$

Thus,

$$f_W(w) = \frac{W+2}{12}, \quad -2 < W < 1.$$

4.2 Case 2: $1 < W < 2$

$$\int_0^{3} \frac{1}{3} \times \frac{1}{4} dx = \frac{1}{4}.$$

Thus,

$$f_W(w) = \frac{1}{4}, \quad 1 < W < 2.$$

4.3 Case 3: $2 < W < 5$

$$\int_{W-2}^{3} \frac{1}{3} \times \frac{1}{4} dx = \frac{5-W}{12}.$$

Thus,

$$f_W(w) = \frac{5-W}{12}, \quad 2 < W < 5.$$

5 Results & Discussion

5.1 Key Insights from the Technique

- **Systematic limit determination**: Instead of case-by-case analysis, the technique **automatically** identifies transition points.
- **Eliminates trial and error**: The method provides a structured way to compute convolution limits **without guesswork**.
- **Universality**: This technique applies to any pair of independent random variables with known bounds.

5.2 Applications

- **Probability Theory**: Simplifies the convolution of distributions.
- **Signal Processing**: Reduces complexity in filtering operations.
- **Machine Learning**: Useful for probabilistic modeling.

6 Conclusion & Future Work

This paper presents a **new technique for identifying integration limits in convolution**, eliminating the need for tedious case-by-case checking. This method improves computational efficiency and can be extended to **multivariate** distributions. Future work will explore **applications in machine learning and numerical methods**.

7 References

- Probability textbooks on convolution methods.
- Signal processing references on numerical integration techniques.
- Research papers on efficient computation of probability densities.